

Irrationality of $\zeta_q(1)$ and $\zeta_q(2)$ [★]

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Abstract

In this paper we show how one can obtain simultaneous rational approximants for $\zeta_q(1)$ and $\zeta_q(2)$ with a common denominator by means of Hermite-Padé approximation using multiple little q -Jacobi polynomials and we show that properties of these rational approximants prove that $1, \zeta_q(1), \zeta_q(2)$ are linearly independent over \mathbb{Q} . In particular this implies that $\zeta_q(1)$ and $\zeta_q(2)$ are irrational. Furthermore we give an upper bound for the measure of irrationality.

Key words: q -zeta function, irrationality, simultaneous rational approximation

1 Introduction

There are a lot of things known about the irrationality of the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at integer values $s \in \{2, 3, 4, \dots\}$. It is known that $\zeta(2n) = (-1)^{n-1} 2^{2n-1} B_{2n} \pi^{2n} / (2n)!$, where B_{2n} are Bernoulli numbers (which are rational), so it follows that $\zeta(2n)$ is an irrational number for $n \in \mathbb{N}_0 = \{1, 2, 3, \dots\}$. It is also known that $\zeta(3)$ is irrational (Apéry [2]) and that $\zeta(2n+1)$ is irrational for an infinite number of $n \in \mathbb{N}_0$ [3] [6] [21]. Furthermore at least one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational [16] [20].

A possible way for a q -extension of the Riemann zeta-function is

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1} q^n}{1 - q^n}, \quad s = 1, 2, \dots, \quad (1.1)$$

[★] This work was supported by INTAS Research Network NeCCA (03-51-6637), FWO project G.0455.04 and OT/04/21 of K.U.Leuven.

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with $q \in \mathbb{C}$ and $|q| < 1$. Then the limit relations

$$\lim_{\substack{q \rightarrow 1 \\ |q| < 1}} (1 - q)^s \zeta_q(s) = (s - 1)! \zeta(s), \quad s = 2, 3, \dots, \quad (1.2)$$

hold [23]. Earlier it was shown that $\zeta_q(1)$ is irrational whenever $q = 1/p$, with p an integer greater than 1 (see, e.g., [18] and the references there). Results of Nesterenko [12] show that $\zeta_q(2)$ is transcendental for every algebraic number q with $0 < |q| < 1$. Zudilin gave an upper bound for the measure of irrationality of $\zeta_q(2)$ [22] with $1/q \in \{2, 3, 4, \dots\}$. Furthermore Krattenthaler, Rivoal and Zudilin [11] proved that there are infinitely many $\zeta_q(2n)$ (and infinitely many $\zeta_q(2n + 1)$) which are irrational, and that at least one of the numbers $\zeta_q(3), \zeta_q(5), \zeta_q(7), \zeta_q(9), \zeta_q(11)$ is irrational whenever $1/q \in \mathbb{Z}$ and $q \neq \pm 1$. From now we only use values of q for which $p = 1/q \in \mathbb{N} \setminus \{0, 1\} = \{2, 3, 4, \dots\}$.

In this paper we will show how one can obtain good *simultaneous* rational approximations for $\zeta_q(1)$ and $\zeta_q(2)$ with a common denominator, which are related to multiple little q -Jacobi polynomials and Hermite-Padé approximation techniques. The method that we use is an extension of the method that Van Assche [18] used to prove the irrationality of $\zeta_q(1)$ and is an application of Hermite-Padé approximation of a system of Markov functions [13, Chapter 4] [19].

In Section 2 we will describe the construction of the simultaneous rational approximants using Hermite-Padé approximation to two Markov functions which are chosen appropriately. The solution of this Hermite-Padé approximation problem depends on multiple little q -Jacobi polynomials [14], and in Section 3 we give the relevant formulas for the special case where all parameters are equal to zero. In Section 4 we will show that these rational approximations give a good rational approximation to $\zeta_q(1)$ and in particular we will prove the following result:

Theorem 1.1 *Suppose $p > 1$ is an integer and $q = 1/p$. Let α_n and β_n be given by (4.3)–(4.4) for all $n \in \mathbb{N}$. Then $\alpha_n, \beta_n \in \mathbb{Z}$ and $\beta_n \zeta_q(2) - \alpha_n \neq 0$. Furthermore*

$$\lim_{n \rightarrow \infty} |\beta_n \zeta_q(1) - \alpha_n|^{1/n^2} \leq p^{-\frac{3(\pi^2 - 4)}{\pi^2}} < 1.$$

In Section 5 we show that we also get a good rational approximation to $\zeta_q(2)$:

Theorem 1.2 *Suppose $p > 1$ is an integer and $q = 1/p$. Let a_n and b_n be given by (5.2)–(5.3) for all $n \in \mathbb{N}$. Then $a_n, b_n \in \mathbb{Z}$ and $b_n \zeta_q(2) - a_n \neq 0$. Furthermore*

$$\lim_{n \rightarrow \infty} |b_n \zeta_q(2) - a_n|^{1/n^2} \leq p^{-\frac{3(\pi^2 - 8)}{\pi^2}} < 1.$$

These rational approximations are good enough to conclude that $\zeta_q(1)$ and $\zeta_q(2)$ are irrational, by using the following elementary lemma:

Lemma 1.1 *Let x be a real number and suppose there exist integers a_n, b_n ($n \in \mathbb{N}$) such that*

- (1) $b_n x - a_n \neq 0$ for every $n \in \mathbb{N}$,
- (2) $\lim_{n \rightarrow \infty} |b_n x - a_n| = 0$.

Then x is irrational.

This lemma expresses the fact that a rational number is approximable to order 1 by rational numbers and to no higher order [9, Theorem 186]. The measure of irrationality $r(x)$ can be defined as

$$r(x) = \inf\{r \in \mathbb{R} : |x - a/b| < 1/b^r \text{ has at most finitely many integer solutions } (a, b)\}.$$

It is known that if $|x - a_n/b_n| = \mathcal{O}(1/b_n^{1+s})$ with $0 < s < 1$ and $b_n < b_{n+1} < b_n^{1+o(1)}$, then the measure of irrationality $r(x)$ satisfies $2 \leq r(x) \leq 1 + 1/s$ (see, e.g., [4, exercise 3 on p. 376] for the upper bound; the lower bound follows since every irrational number is approximable to order 2 [9, Theorem 187]).

Our rational approximations to $\zeta_q(1)$ give the upper bound $5.0444\dots$ for the measure of irrationality of $\zeta_q(1)$, which is not as good as the upper bound of $2.5082\dots$ in [18] or the upper bound of $2.4234\dots$ in [23]. For $\zeta_q(2)$ we obtain the upper bound $15.8369\dots$ for the measure of irrationality, which is not as good as the upper bound $4.07869374\dots$ that Zudilin obtained in [22]. Our rational approximations, however, use the same denominator and hence we have constructed simultaneous rational approximants. The price we pay for this is that the order of approximation for each individual number is not as good as possible. But we gain some very important information because our simultaneous rational approximants are good enough to prove our main result:

Theorem 1.3 *The numbers $1, \zeta_q(1)$ and $\zeta_q(2)$ are linearly independent over \mathbb{Q} .*

This result is stronger than the statement that $\zeta_q(1)$ and $\zeta_q(2)$ are irrational. We prove this theorem by means of the following lemma, which extends Lemma 1.1 (see [10, Lemma 2.1] for a similar lemma but with a different first condition).

Lemma 1.2 *Let x, y be real numbers. Suppose that for all $(a, b, c) \in \mathbb{Z}^3 \setminus (0, 0, 0)$ and for infinity many positive integers $n \in \Lambda \subset \mathbb{N}$, there exist integers p_n, q_n and r_n such that the following three conditions are satisfied:*

- (1) $ap_n + bq_n + cr_n \neq 0$ for every $n \in \Lambda$,
- (2) $|p_n x - q_n| \rightarrow 0$ as $n \rightarrow \infty$,
- (3) $|p_n y - r_n| \rightarrow 0$ as $n \rightarrow \infty$.

Then $1, x, y$ are linearly independent over \mathbb{Q} .

Proof. Suppose that $1, x$ and y are linearly dependent over \mathbb{Q} . Then there exist integers

$a_1, a_2, b_1, b_2, c_1, c_2$ such that

$$\frac{a_1}{a_2} + \frac{b_1}{b_2}x + \frac{c_1}{c_2}y = 0.$$

When we multiply this by $a_2b_2c_2$ we get $a_1b_2c_2 + a_2b_1c_2x + a_2b_2c_1y = 0$, hence there exist integers a, b, c such that $a + bx + cy = 0$. Multiplying by p_n and adding the terms $-bq_n - cr_n$ on both sides, we obtain

$$b(p_nx - q_n) + c(p_ny - r_n) = -(ap_n + bq_n + cr_n).$$

On the right hand side we have an integer different from zero, hence the right hand side is in absolute value at least 1 for every $n \in \Lambda$. But the expression on the left hand side tends to 0 as n tends to infinity, hence we have a contradiction. \square

If we take $x = \zeta_q(1)$ and $y = \zeta_q(2)$, then Condition (2) follows from Theorem 1.1 and Condition (3) follows from Theorem 1.2. In Section 6 we will show that Condition (1) of Lemma 1.2 holds. This requires some results about cyclotomic polynomials and cyclotomic numbers from number theory.

2 Multiple little q -Jacobi polynomials

2.1 Multiple little q -Jacobi polynomials

Little q -Jacobi polynomials are orthogonal polynomials on the exponential lattice $\{q^k, k = 0, 1, 2, \dots\}$, where $0 < q < 1$. In order to express the orthogonality relations, we will use the q -integral

$$\int_0^1 f(x) d_q x = \sum_{k=0}^{\infty} q^k f(q^k), \quad (2.1)$$

(see, e.g., [1, §10.1], [8, §1.11]) where f is a function on $[0, 1]$ which is continuous at 0. The orthogonality is given by

$$\int_0^1 p_n(x; \alpha, \beta|q) x^k w(x; \alpha, \beta|q) d_q x = 0, \quad k = 0, 1, \dots, n-1, \quad (2.2)$$

where

$$w(x; a, b|q) = \frac{(qx; q)_{\infty}}{(q^{\beta+1}x; q)_{\infty}} x^{\alpha}. \quad (2.3)$$

We have used the notation

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).$$

In order that the q -integral of w is finite, we need to impose the restrictions $\alpha, \beta > -1$. The orthogonality conditions (2.2) determine the polynomials $p_n(x; \alpha, \beta|q)$ up to a

multiplicative factor. When we define $p_n(x; \alpha, \beta|q)$ as monic polynomials, they are uniquely determined by the orthogonality conditions.

The Rodrigues formula for $p_n(x; \alpha, \beta|q)$ is given by

$$p_n(x; \alpha, \beta|q) = \frac{(q^{\alpha+n} - q^{\alpha+n-1})^n}{(q^{\alpha+\beta+n+1}; q)_n w(x, \alpha, \beta|q)} D_{q^{-1}}^n w(x, \alpha + n, \beta + n|q) \quad (2.4)$$

and an explicit formula by

$$\begin{aligned} p_n(x; \alpha, \beta|q) &= (-1)^n q^{n(n-1)/2} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix} \middle| q; qx \right) \\ &= (-1)^n q^{n(n-1)/2} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{n+\alpha+\beta+1}; q)_k}{(q^{\alpha+1}; q)_k} \frac{q^k x^k}{(q; q)_k}. \end{aligned} \quad (2.5)$$

We have introduced multiple little q -Jacobi polynomials in [14]. Suppose that $\beta > -1$ and that $\alpha_1, \alpha_2, \dots, \alpha_r$ are such that each $\alpha_i > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. Then the multiple little q -Jacobi polynomial $p_{\vec{n}}(x; \vec{\alpha}, \beta|q)$ for the multi-index $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ is the monic polynomial of degree $|\vec{n}| = n_1 + n_2 + \dots + n_r$ that satisfies the orthogonality conditions

$$\int_0^1 p_{\vec{n}}(x; \vec{\alpha}, \beta|q) x^k w(x; \alpha_j, \beta|q) d_q x = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2, \dots, r, \quad (2.6)$$

These are the multiple little q -Jacobi polynomials of the first kind. We only consider the case $r = 2$, so we take $\vec{n} = (n, m)$, with $m \leq n$. From [14] we know that the Rodrigues formula for $p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q)$ is given by

$$\begin{aligned} p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) &= \frac{(q^{\alpha_1+n+m} - q^{\alpha_1+n+m-1})^n (q^{\alpha_2+m} - q^{\alpha_2+m-1})^m}{(q^{\beta+\alpha_1+n+m+1}; q)_n (q^{\beta+\alpha_2+n+m+1}; q)_m w(x, \alpha_1, \beta|q)} \\ &\quad \times D_{q^{-1}}^n [x^{\alpha_1-\alpha_2+n} D_{q^{-1}}^m w(x, \alpha_2 + m, \beta + n + m|q)] \end{aligned} \quad (2.7)$$

and that an explicit expression for $p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q)$ is given by

$$\begin{aligned} p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) &= (-1)^{n+m} \frac{q^{nm} (q^{\alpha_1+1}; q)_n (q^{\alpha_2+1}; q)_m q^{n(n-1)/2} q^{m(m-1)/2}}{(q^{\beta+\alpha_1+n+m+1}; q)_n (q^{\beta+\alpha_2+n+m+1}; q)_m} \\ &\quad \times \sum_{k=0}^m \sum_{j=0}^n \frac{(q^{-n}; q)_j (q^{-m}; q)_k (q^{\alpha_1+n+1}; q)_k (q^{\alpha_1+\beta+n+1}; q)_{j+k} (q^{\alpha_2+\beta+n+m+1}; q)_k}{(q^{\alpha_1+1}; q)_{j+k} (q^{\alpha_1+\beta+n+1}; q)_k (q^{\alpha_2+1}; q)_k} \\ &\quad \times \frac{q^{k+j} x^{k+j}}{q^{kn} (q; q)_k (q; q)_j}. \end{aligned} \quad (2.8)$$

2.2 The case $\alpha_1 = \alpha_2$

When we take $\alpha_1 = \alpha_2 = \alpha$, then (2.6) gives only n conditions. However, when we look at expression (2.8) we see that $p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q)$ still has degree $n + m$. We use the notation $p_{n,m}^{(\alpha, \alpha, \beta)}(x)$ for $p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q)$ with $\alpha_1 = \alpha_2 = \alpha$. We will now show that $p_{n,m}^{(\alpha, \alpha, \beta)}(x)$, defined by the Rodrigues formula (2.7), again satisfies $n + m$ orthogonality conditions, namely

$$\int_0^1 p_{n,m}^{(\alpha, \alpha, \beta)}(x) w(x, \alpha, \beta|q) x^\ell d_q x = 0, \quad \ell = 0, 1, \dots, n-1, \quad (2.9)$$

$$\int_0^1 p_{n,m}^{(\alpha, \alpha, \beta)}(x) w(x, \alpha, \beta|q) x^\ell \log_q x d_q x = 0, \quad \ell = 0, 1, \dots, m-1. \quad (2.10)$$

To prove these orthogonality conditions we will need the following lemma.

Lemma 2.1 (Summation by parts) *When $g(q^{-1}) = 0$ we have that*

$$\sum_{k=0}^{\infty} q^k f(q^k) D_{q^{-1}} g(x)|_{q^k} = -q \sum_{k=0}^{\infty} q^k g(q^k) D_q f(x)|_{q^k}. \quad (2.11)$$

We start with the first n orthogonality conditions. When we use the Rodrigues formula (2.7), we obtain

$$\int_0^1 p_{n,m}^{(\alpha, \alpha, \beta)}(x) w(x, \alpha, \beta|q) x^\ell d_q x = C_{n,m} \int_0^1 D_{q^{-1}}^n [x^n D_{q^{-1}}^m w(x, \alpha + m, \beta + n + m|q)] x^\ell d_q x,$$

with

$$C_{n,m} = \frac{(q^{\alpha+n+m} - q^{\alpha+n+m-1})^n (q^{\alpha+m} - q^{\alpha+m-1})^m}{(q^{\beta+\alpha+n+m+1}; q)_n (q^{\beta+\alpha+n+m+1}; q)_m}.$$

Now we can rewrite the right hand side as an infinite sum using (2.1) and then apply summation by parts (2.11) n times. This gives

$$\begin{aligned} & \int_0^1 p_{n,m}^{(\alpha, \alpha, \beta)}(x) w(x, \alpha, \beta|q) x^\ell d_q x \\ &= (-1)^n q^n C_{n,m} \sum_{k=0}^{\infty} q^k \left[x^n D_{q^{-1}}^m w(x, \alpha + m, \beta + n + m|q) \right] \Big|_{x=q^k} D_q^n x^\ell \Big|_{x=q^k}. \end{aligned}$$

Since $D_q^n x^\ell = 0$ for $\ell < n$, we see that (2.9) holds.

Now we prove (2.10). Again using the Rodrigues formula (2.7) and applying summation by parts (2.11) n times

$$\begin{aligned} & \int_0^1 p_{n,m}^{(\alpha, \alpha, \beta)}(x) w(x, \alpha, \beta|q) x^\ell \log_q x d_q x \\ &= (-1)^n q^n C_{n,m} \int_0^1 D_q^n (x^\ell \log_q x) [x^n D_{q^{-1}}^m w(x, \alpha + m, \beta + n + m|q)] d_q x. \end{aligned}$$

To calculate $D_q^n (x^\ell \log_q x)$ we can use the q -analogue of Leibniz' rule

$$D_q^n [f(x)g(x)] = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} f) (q^k x) (D_q^k g) (x), \quad n = 0, 1, 2, \dots \quad (2.12)$$

This gives

$$\begin{aligned} & \int_0^1 p_{n,m}^{(\alpha,\alpha,\beta)}(x) w(x, \alpha, \beta|q) x^\ell \log_q x d_q x \\ &= C \int_0^1 \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (D_q^{n-k} x^\ell) \Big|_{q^k x} (D_q^k \log_q x) \Big|_x [x^n D_{q^{-1}}^m w(x, \alpha + m, \beta + n + m|q)] d_q x, \end{aligned}$$

with C a constant. Obviously we have for $k < n - \ell$ that $D_q^{n-k} x^\ell = 0$ and for $k \geq n - \ell$ that $D_q^{n-k} x^\ell$ is a polynomial of degree $\ell - n + k$, say $\pi_{k,\ell-n+k}$. It is also easy to see that $D_q^k \log_q x$ is a constant times $1/x^k$ for $k \geq 1$. So we get

$$\begin{aligned} & \int_0^1 p_{n,m}^{(\alpha,\alpha,\beta)}(x) w(x, \alpha, \beta|q) x^\ell \log_q x d_q x \\ &= \int_0^1 \sum_{k=n-\ell}^n c_k \begin{bmatrix} n \\ k \end{bmatrix}_q \pi_{k,\ell-n+k}(q^k x) \frac{1}{x^k} x^n D_{q^{-1}}^m w(x, \alpha + m, \beta + n + m|q) d_q x, \end{aligned}$$

where the c_k are constants. Now $\pi_{k,\ell-n+k}(q^k x) x^n / x^k$ is a polynomial of degree ℓ , say $\rho_{k,\ell}(x)$. Using the Rodrigues formula (2.4) for little q -Jacobi polynomials, we get

$$\begin{aligned} & \int_0^1 p_{n,m}^{(\alpha,\alpha,\beta)}(x) w(x, \alpha, \beta|q) x^\ell \log_q x d_q x \\ &= (-1)^n \sum_{k=n-\ell}^n c'_k \begin{bmatrix} n \\ k \end{bmatrix}_q \int_0^1 \rho_{k,\ell}(x) w(x, \alpha, \beta + n|q) p_m^{(\alpha,\beta+n)}(x) d_q x, \end{aligned}$$

where the c'_k are constants. The orthogonality relations of the little q -Jacobi polynomial imply that the right hand side is 0 for $\ell < m$, so that (2.10) holds.

2.3 The case $\alpha_1 = \alpha_2 = \beta = 0$

From now, we will work with the multiple little q -Jacobi polynomial $p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q)$ where $\alpha_1 = \alpha_2 = \beta = 0$ and with another normalisation, namely

$$p_{n,m}(0; (\alpha_1, \alpha_2), \beta|q) = 1. \quad (2.13)$$

We will denote this polynomial by $p_{n,m}(x)$. Since $w(x, 0, 0|q) = 1$, it follows from (2.9)–(2.10) that $p_{n,m}$ can be defined by

$$\int_0^1 p_{n,m}(x) x^\ell d_q x = 0, \quad \ell = 0, 1, \dots, n-1, \quad (2.14)$$

$$\int_0^1 p_{n,m}(x) x^\ell \log_q x d_q x = 0, \quad \ell = 0, 1, \dots, m-1, \quad (2.15)$$

together with the normalisation (2.13).

From (2.8) we see that

$$p_{n,m}(0; (\alpha_1, \alpha_2), \beta|q) = (-1)^{n+m} \frac{q^{nm} (q^{\alpha_1+1}; q)_n (q^{\alpha_2+1}; q)_m q^{n(n-1)/2} q^{m(m-1)/2}}{(q^{\beta+\alpha_1+n+m+1}; q)_n (q^{\beta+\alpha_2+n+m+1}; q)_m}. \quad (2.16)$$

Multiplying (2.7) and (2.8) by $[p_{n,m}(0; (\alpha_1, \alpha_2), \beta|q)]^{-1}$ and setting $\alpha_1 = \alpha_2 = \beta = 0$ gives

$$p_{n,m}(x) = \sum_{k=0}^m \sum_{j=0}^n \frac{(qx)^{k+j}}{(q; q)_k (q; q)_j} \frac{1}{q^{kn}} \frac{(q^{n+1}; q)_{j+k}}{(q; q)_{j+k}} \frac{(q^{n+m+1}; q)_k}{(q; q)_k} (q^{-n}; q)_j (q^{-m}; q)_k \quad (2.17)$$

and the Rodrigues formula becomes

$$p_{n,m}(x) = (-1)^{n+m} q^{n(n-1)/2} q^{m(m-1)/2} \frac{(q-1)^{n+m}}{(q; q)_n (q; q)_m} D_{q^{-1}}^n [x^n D_{q^{-1}}^m ((qx; q)_{n+m} x^m)]. \quad (2.18)$$

Equation (2.17) expresses the polynomial $p_{n,m}$ in the basis $\{1, x, x^2, \dots, x^{n+m}\}$. Sometimes it is more convenient to use the basis $\{(qx; q)_\ell, 0 \leq \ell \leq n+m\}$. The Rodrigues formula (2.18) allows us to obtain an expression for $p_{n,m}(x)$ in this basis. Recall the q -analog of Newton's binomial formula

$$(x; q)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} (-x)^k, \quad (2.19)$$

and its dual

$$x^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+k(k+1)/2} (x; q)_k. \quad (2.20)$$

This is a special case of the q -binomial series [1, §10.2] [8, §1.3]

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |q| < 1, |x| < 1. \quad (2.21)$$

Using (2.20) with argument $q^{n+m+1}x$ and exponent m gives

$$x^m = p^{(n+m+1)m} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{-mk+k(k+1)/2} (q^{n+m+1}x; q)_k.$$

Using this in the Rodrigues formula (2.18), we find

$$p_{n,m}(x) = (-1)^{n+m} q^{n(n-1)/2} q^{m(m-1)/2} \frac{(q-1)^{n+m}}{(q;q)_n (q;q)_m} p^{(n+m+1)m} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{-mk+k(k+1)/2} D_{q^{-1}}^n [x^n D_{q^{-1}}^m (qx; q)_{n+m+k}],$$

because $(qx; q)_{n+m} (q^{n+m+1}x; q)_k = (qx; q)_{n+m+k}$. One easily finds that

$$D_p^k (qx; q)_n = \frac{(q; q)_n}{(q; q)_{n-k} (1-p)^k} (qx; q)_{n-k}, \quad (2.22)$$

so we get

$$p_{n,m}(x) = (-1)^{n+m} q^{n(n-1)/2} q^{m(m-1)/2} \frac{(q-1)^{n+m}}{(q;q)_n (q;q)_m} p^{(n+m+1)m} \times \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{-mk+k(k+1)/2} \frac{(q; q)_{n+m+k}}{(q; q)_{n+k} (1-p)^m} D_{q^{-1}}^n [x^n (qx; q)_{n+k}].$$

Using (2.20) again, this time with argument $q^{n+k+1}x$ and exponent n , gives

$$x^n = p^{(n+k+1)n} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{-nj+j(j+1)/2} (q^{n+k+1}x; q)_j,$$

and since $(qx; q)_{n+k} (q^{n+k+1}x; q)_j = (qx; q)_{n+k+j}$ we obtain

$$p_{n,m}(x) = (-1)^{n+m} q^{n(n-1)/2} q^{m(m-1)/2} \frac{(q-1)^{n+m}}{(q;q)_n (q;q)_m} p^{(n+m+1)m} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-1)^k q^{-mk+k(k+1)/2} \times \frac{(q; q)_{n+m+k}}{(q; q)_{n+k} (1-p)^m} p^{(n+k+1)n} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{-nj+j(j+1)/2} D_{q^{-1}}^n (qx; q)_{n+k+j}.$$

Again using formula (2.22) and also using that

$$(q; q)_k = (-1)^k p^{-k(k+1)/2} (p; p)_k, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = p^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_p,$$

we can rewrite the expression for $p_{n,m}(x)$ as

$$p_{n,m}(x) = (-1)^{n+m} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \times (qx; q)_{k+j} p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2}. \quad (2.23)$$

3 Hermite-Padé approximation of f_1 and f_2

We define two measures μ_1 and μ_2 by taking $d\mu_1(x) = d_q x$ and $d\mu_2(x) = \log_q(x) d_q x$, where d_q is defined by (2.1). Then μ_1 and μ_2 are supported on $\{q^k, k = 0, 1, 2, \dots\}$, which is a bounded set in $[0, 1]$ with one accumulation point at 0. The Markov functions for the measures μ_1 and μ_2 are

$$f_1(z) = \int_0^1 \frac{d\mu_1(x)}{z - x} = \sum_{k=0}^{\infty} \frac{q^k}{z - q^k} \quad (3.1)$$

$$f_2(z) = \int_0^1 \frac{d\mu_2(x)}{z - x} = \sum_{k=0}^{\infty} \frac{kq^k}{z - q^k}. \quad (3.2)$$

Observe that for every $N \in \mathbb{N}$

$$\begin{aligned} \zeta_q(1) &= f_1(p^N) + \sum_{k=1}^{N-1} \frac{1}{p^k - 1}, \\ \zeta_q(2) &= f_2(p^N) + \sum_{k=1}^{N-1} \frac{k}{p^k - 1} + Nf_1(p^N), \end{aligned}$$

therefore we will look for rational approximants of $f_1(z)$ and $f_2(z)$ with common denominator and evaluate these at $z = p^N$ for an appropriate choice of N . We can find these rational approximants by Hermite-Padé approximation of type II.

For Hermite-Padé approximation of type II one requires a polynomial $p_{n,m}$ of degree $\leq n + m$, and polynomials $q_{n,m}$ and $r_{n,m}$ such that

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad z \rightarrow \infty, \quad (3.3)$$

$$p_{n,m}(z)f_2(z) - r_{n,m}(z) = \mathcal{O}\left(\frac{1}{z^{m+1}}\right). \quad z \rightarrow \infty, \quad (3.4)$$

It is known [13, Chapter 4] that for $m \leq n$ the polynomial $p_{n,m}$ is, up to a multiplicative factor, uniquely given by

$$\int_0^1 p_{n,m}(x) x^\ell d\mu_1(x) = 0, \quad \ell = 0, 1, \dots, n-1, \quad (3.5)$$

$$\int_0^1 p_{n,m}(x) x^\ell d\mu_2(x) = 0, \quad \ell = 0, 1, \dots, m-1, \quad (3.6)$$

and that $q_{n,m}$ and $r_{n,m}$ are given by

$$q_{n,m}(z) = \int_0^1 \frac{p_{n,m}(z) - p_{n,m}(x)}{z - x} d\mu_1(x), \quad r_{n,m}(z) = \int_0^1 \frac{p_{n,m}(z) - p_{n,m}(x)}{z - x} d\mu_2(x). \quad (3.7)$$

The remainder in the approximation (3.3) is

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = \int_0^1 \frac{p_{n,m}(x)}{z-x} d\mu_1(x), \quad (3.8)$$

and for (3.4)

$$p_{n,m}(z)f_2(z) - r_{n,m}(z) = \int_0^1 \frac{p_{n,m}(x)}{z-x} d\mu_2(x). \quad (3.9)$$

Comparing (3.5)–(3.6) with (2.14)–(2.15) we see that the common denominator is given by the multiple little q -Jacobi polynomial $p_{n,m}$. So (2.23) gives an explicit expression for $p_{n,m}$. Then it follows that we can compute $q_{n,m}$ and $r_{n,m}$ explicitly using (3.7). For $q_{n,m}$ we have to compute

$$q_{n,m}(z) = \sum_{\ell=0}^{\infty} \frac{p_{n,m}(z) - p_{n,m}(q^\ell)}{z - q^\ell} q^\ell. \quad (3.10)$$

By using the explicit expression (2.23) for $p_{n,m}(z)$, we find

$$\begin{aligned} q_{n,m}(z) &= (-1)^{n+m} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ &\quad \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{\ell=0}^{\infty} \frac{(qz; q)_{k+j} - (q^{\ell+1}; q)_{k+j}}{z - q^\ell} q^\ell. \end{aligned}$$

Now use

$$\frac{(qx; q)_k - (qy; q)_k}{x - y} = - \sum_{\ell=1}^k q^\ell (qy; q)_{\ell-1} (q^{\ell+1}x; q)_{k-\ell},$$

which one can prove by induction, then this gives

$$\begin{aligned} q_{n,m}(z) &= (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ &\quad \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} q^r (q^{r+1}z; q)_{k-r+j} \sum_{\ell=0}^{\infty} q^\ell (q^{\ell+1}; q)_{r-1}. \end{aligned}$$

By using the q -binomial series (2.21), we can compute the modified moments

$$\sum_{\ell=0}^{\infty} q^\ell (q^{\ell+1}; q)_{r-1} = \frac{1}{1 - q^r}, \quad (3.11)$$

so that

$$\begin{aligned} q_{n,m}(z) &= (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ &\quad \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} \frac{(q^{r+1}z; q)_{k-r+j}}{p^r - 1}. \end{aligned} \quad (3.12)$$

For an explicit expression of $r_{n,m}$ we use (3.7), which gives

$$r_{n,m}(z) = \sum_{\ell=0}^{\infty} \frac{p_{n,m}(z) - p_{n,m}(q^\ell)}{z - q^\ell} \ell q^\ell. \quad (3.13)$$

Completely analogous to $q_{n,m}$ we find

$$\begin{aligned} r_{n,m}(z) &= (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ &\quad \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} q^r (q^{r+1}z; q)_{k-r+j} \sum_{\ell=0}^{\infty} \ell q^\ell (q^{\ell+1}; q)_{r-1}. \end{aligned}$$

Now we have more work to compute

$$\sum_{\ell=0}^{\infty} \ell q^\ell (q^{\ell+1}; q)_{r-1} = (q; q)_{r-1} \sum_{\ell=0}^{\infty} \ell q^\ell \frac{(q^r; q)_\ell}{(q; q)_\ell}. \quad (3.14)$$

Using the q -binomial series (2.21), we have that

$$(x; q)_r \sum_{\ell=0}^{\infty} \frac{(q^r; q)_\ell}{(q; q)_\ell} x^\ell = 1.$$

By taking the derivative with respect to x we find

$$\frac{d}{dx} [(x; q)_r] \sum_{\ell=0}^{\infty} \frac{(q^r; q)_\ell}{(q; q)_\ell} x^\ell + (x; q)_r \sum_{\ell=0}^{\infty} \ell \frac{(q^r; q)_\ell}{(q; q)_\ell} x^{\ell-1} = 0.$$

It is not difficult to see that

$$\frac{d}{dx} [(x; q)_r] = - \sum_{i=0}^{r-1} \frac{(x; q)_r}{1 - xq^i} q^i,$$

so we find

$$\sum_{\ell=0}^{\infty} \ell \frac{(q^r; q)_\ell}{(q; q)_\ell} x^\ell = x \sum_{i=0}^{r-1} \frac{q^i}{1 - xq^i} \sum_{\ell=0}^{\infty} \frac{(q^r; q)_\ell}{(q; q)_\ell} x^\ell.$$

From this it follows that (3.14) becomes

$$\sum_{\ell=0}^{\infty} \ell q^\ell (q^{\ell+1}; q)_{r-1} = (q; q)_{r-1} q \sum_{i=0}^{r-1} \frac{q^i}{1 - q^{i+1}} \sum_{\ell=0}^{\infty} \frac{(q^r; q)_\ell}{(q; q)_\ell} q^\ell.$$

Using expression (3.11) for the modified moments we find

$$\sum_{\ell=0}^{\infty} \ell q^\ell (q^{\ell+1}; q)_{r-1} = \frac{1}{1 - q^r} \sum_{i=1}^r \frac{1}{p^i - 1}.$$

When we use this in the expression for $r_{n,m}$ we get

$$r_{n,m}(z) = (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} \sum_{i=1}^r \frac{(q^{r+1}z; q)_{k-r+j}}{(p^r - 1)(p^i - 1)}. \quad (3.15)$$

We will evaluate these functions $p_{n,m}(z)$, $q_{n,m}(z)$, $r_{n,m}(z)$ at $z = p^{n+m}$. First we will show that $p_{n,m}(p^{n+m})$ is an integer. From the q -version of Pascal's triangle identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_p = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_p = \begin{bmatrix} n-1 \\ k \end{bmatrix}_p + p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p, \quad (3.16)$$

it follows (by induction) that $\begin{bmatrix} n \\ k \end{bmatrix}_p$ is an integer whenever p is an integer. Furthermore we have

$$(qp^{n+m}; q)_{k+j} = (p^{n+m-1}; p^{-1})_{k+j} = \prod_{i=1}^{k+j} (1 - p^{n+m-i}).$$

For each value of i in this product, the factor $1 - p^{n+m-i}$ is an integer. This means that $p_{n,m}(p^{n+m})$, with $p_{n,m}$ given by (2.23), is an integer.

Now we will evaluate $q_{n,m}(z)$ and $r_{n,m}(z)$ at p^{n+m} . We can use (3.12) at $z = p^{n+m}$ and $(q^{r+1}p^{n+m}; q)_{k-r+j} = (p^{n+m-k-j}; p)_{k-r+j}$, to find

$$q_{n,m}(p^{n+m}) = (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} \frac{(p^{n+m-k-j}; p)_{k-r+j}}{p^r - 1}. \quad (3.17)$$

The terms in the sum for $q_{n,m}(p^{n+m})$ are not all integers, because of the expression $p^r - 1$ in the denominators. In order to obtain an integer we have to multiply $q_{n,m}(p^{n+m})$ by a multiple of all $p^r - 1$ for $r = 1, 2, \dots, n+m$.

For $r_{n,m}(p^{n+m})$ we use (3.15) at $z = p^{n+m}$ and again $(q^{r+1}p^{n+m}; q)_{k-r+j} = (p^{n+m-k-j}; p)_{k-r+j}$, to find

$$r_{n,m}(p^{n+m}) = (-1)^{n+m+1} \sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \begin{bmatrix} n+m+k \\ m \end{bmatrix}_p \begin{bmatrix} n+k+j \\ n \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \\ \times p^{(n-j)(n-j+1)/2} p^{(m-k)(m-k+1)/2} \sum_{r=1}^{k+j} \sum_{i=1}^r \frac{(p^{n+m-k-j}; p)_{k-r+j}}{(p^r - 1)(p^i - 1)}. \quad (3.18)$$

We see that in order to get an integer now, we have to multiply $r_{n,m}(p^{n+m})$ by a multiple of $(p^r - 1)(p^i - 1)$ for $r = 1, 2, \dots, n+m$ and $i = 1, 2, \dots, n+m$.

Define

$$d_n(x) = \prod_{k=1}^n \Phi_k(x), \quad (3.19)$$

where

$$\Phi_n(x) = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^n (x - e^{2\pi i k/n}) \quad (3.20)$$

are the cyclotomic polynomials. Each cyclotomic polynomial is monic and has integer coefficients. It is known [17] that

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \quad (3.21)$$

and that every cyclotomic polynomial is irreducible over $\mathbb{Q}[x]$. Hence $d_n(p)$ is a multiple of all $p^\ell - 1$ for $\ell = 1, 2, \dots, n$. The growth of this sequence is given by the following lemma [18, Lemma 2].

Lemma 3.1 *Suppose $x > 1$ and let $d_n(x)$ be given by (3.19). Then*

$$\lim_{n \rightarrow \infty} d_n(x)^{1/n^2} = x^{3/\pi^2}.$$

Since $d_{n+m}(p)$ is a multiple of $p^\ell - 1$ for all $\ell = 1, 2, \dots, n+m$, we conclude that $d_{n+m}(p)q_{n,m}(p^{n+m})$ and $d_{n+m}^2(p)r_{n,m}(p^{n+m})$ are integers.

4 Rational approximations to $\zeta_q(1)$

From (1.1) we know that

$$\zeta_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{1}{p^n - 1}.$$

In this section we will construct rational approximations to $\zeta_q(1)$, prove the irrationality and give an upper bound for its measure of irrationality.

4.1 Rational approximations

In (3.1) we defined the function f_1 . When we evaluate f_1 at p^{n+m} , we get

$$f_1(p^{n+m}) = \sum_{k=0}^{\infty} \frac{q^k}{p^{n+m} - q^k} = \sum_{k=0}^{\infty} \frac{1}{p^{n+m+k} - 1} = \zeta_q(1) - \sum_{k=1}^{n+m-1} \frac{1}{p^k - 1} \quad (4.1)$$

and hence

$$\zeta_q(1) = f_1(p^{n+m}) + \sum_{k=1}^{n+m-1} \frac{1}{p^k - 1}. \quad (4.2)$$

We now take $m = n - 1$ and define

$$\beta_n = d_{2n-1}(p) p_{n,n-1}(p^{2n-1}), \quad (4.3)$$

$$\alpha_n = d_{2n-1}(p) \left[q_{n,n-1}(p^{2n-1}) + p_{n,n-1}(p^{2n-1}) \sum_{k=1}^{2n-2} \frac{1}{p^k - 1} \right]. \quad (4.4)$$

Then it follows from equation (4.2) and from the Hermite-Padé approximation of f_1 (3.8) that

$$\begin{aligned} \beta_n \zeta_q(1) - \alpha_n &= d_{2n-1}(p) [p_{n,n-1}(p^{2n-1}) f_1(p^{2n-1}) - q_{n,n-1}(p^{2n-1})] \\ &= d_{2n-1}(p) \int_0^1 \frac{p_{n,n-1}(x)}{p^{2n-1} - x} d_q x. \end{aligned} \quad (4.5)$$

4.2 Irrationality of $\zeta_q(1)$

Using Lemma 1.1, we can prove the irrationality of $\zeta_q(1)$. We will first show that

$$\lim_{n \rightarrow \infty} |\beta_n \zeta_q(1) - \alpha_n| = 0,$$

i.e., using (4.5), we want to find an estimate for

$$\int_0^1 \frac{p_{n,n-1}(x)}{z - x} d_q x.$$

When we use the Rodrigues formula (2.18) for $p_{n,m}$ we get

$$\int_0^1 \frac{p_{n,m}(x)}{z - x} d_q x = C \int_0^1 D_{q^{-1}}^n [x^n D_{q^{-1}}^m ((qx; q)_{n+m} x^m)] \frac{1}{z - x} d_q x,$$

where

$$C = (-1)^{n+m} q^{n(n-1)/2} q^{m(m-1)/2} \frac{(q-1)^{n+m}}{(q; q)_n (q; q)_m}. \quad (4.6)$$

Repeated application of summation by parts (2.11) gives

$$\int_0^1 \frac{p_{n,m}(x)}{z - x} d_q x = C (-q)^n \int_0^1 x^n D_{q^{-1}}^m ((qx; q)_{n+m} x^m) D_q^n \left[\frac{1}{z - x} \right] d_q x.$$

Now it is easy to see by induction that

$$D_q^n \left[\frac{1}{z - x} \right] = \frac{(q; q)_n (1 - q)^{-n}}{(z - x)(z - xq) \dots (z - xq^n)} = \frac{(q; q)_n}{(1 - q)^n z^{n+1}} \frac{1}{(\frac{x}{z}; q)_{n+1}},$$

so for $z = p^{n+m}$ we find

$$\int_0^1 \frac{p_{n,m}(x)}{p^{n+m} - x} d_q x = C(-q)^n \frac{(q; q)_n}{(1-q)^n p^{(n+m)(n+1)}} \int_0^1 \frac{x^n D_{q^{-1}}^m ((qx; q)_{n+m} x^m)}{(q^{n+m} x; q)_{n+1}} d_q x.$$

Again we apply summation by parts (2.11) several times and use the q -analogue of Leibniz' formula (2.12). This gives

$$\begin{aligned} \int_0^1 \frac{p_{n,m}(x)}{p^{n+m} - x} d_q x &= C(-q)^{n+m} \frac{(q; q)_n}{(1-q)^n p^{(n+m)(n+1)}} \\ &\quad \times \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q \int_0^1 (qx; q)_{n+m} x^m D_q^{m-k} (x^n) \Big|_{q^k x} D_q^k \frac{1}{(q^{n+m} x; q)_{n+1}} d_q x. \end{aligned}$$

By induction, it is easy to see that for $r \leq s$

$$D_q^r (x^s) = \frac{(q; q)_s x^{s-r}}{(1-q)^r (q; q)_{s-r}}, \quad (4.7)$$

and that

$$D_q^k \frac{1}{(q^{n+m} x; q)_{n+1}} = \frac{q^{k(n+m)} (q^{n+1}; q)_k}{(1-q)^k (q^{n+m} x; q)_{n+k+1}}.$$

Using this we find

$$\begin{aligned} \int_0^1 \frac{p_{n,m}(x)}{p^{n+m} - x} d_q x &= C(-q)^{n+m} \frac{(q; q)_n}{(1-q)^n p^{(n+m)(n+1)}} \\ &\quad \times \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{q^{k(2n+k)} (q; q)_n (q^{n+1}; q)_k}{(1-q)^m (q; q)_{n-m+k}} \int_0^1 \frac{(qx; q)_{n+m} x^{n+k}}{(q^{n+m} x; q)_{n+k+1}} d_q x. \end{aligned} \quad (4.8)$$

The integrand is always positive and $(q^{n+m} x; q)_{n+k+1} \geq (q^{n+m}; q)_{n+k+1}$, hence we find

$$\begin{aligned} \int_0^1 \frac{(qx; q)_{n+m} x^{n+k}}{(q^{n+m} x; q)_{n+k+1}} d_q x &\leq \frac{1}{(q^{n+m}; q)_{n+k+1}} \int_0^1 (qx; q)_{n+m} x^{n+k} d_q x \\ &\leq \frac{1}{(q^{n+m}; q)_{n+k+1}} \sum_{\ell=0}^{\infty} (q^{\ell+1}; q)_{n+m} (q^{\ell})^{n+k+1} \\ &\leq \frac{(q; q)_{n+m}}{(q^{n+m}; q)_{n+k+1}} \sum_{\ell=0}^{\infty} \frac{(q^{n+m+1}; q)_{\ell}}{(q; q)_{\ell}} (q^{n+k+1})^{\ell}. \end{aligned}$$

Using the q -binomial series (2.21), we then find

$$\begin{aligned} \int_0^1 \frac{(qx; q)_{n+m} x^{n+k}}{(q^{n+m} x; q)_{n+k+1}} d_q x &\leq \frac{(q; q)_{n+m}}{(q^{n+m}; q)_{n+k+1}} \frac{(q^{2n+k+m+2}; q)_{\infty}}{(q^{n+k+1}; q)_{\infty}} \\ &= \frac{(q; q)_{n+k}}{(q^{n+m}; q)_{n+k+1} (q^{n+m+1}; q)_{n+k+1}}. \end{aligned}$$

This estimate, together with the definition of C in (4.6), then gives

$$\left| \int_0^1 \frac{p_{n,m}(x)}{p^{n+m} - x} d_q x \right| \leq \frac{q^{n(n-1)/2} q^{m(m-1)/2} q^{n+m}}{p^{(n+m)(n+1)} (q; q)_m} \times \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{q^{k(2n+k)} (q; q)_n (q^{n+1}; q)_k (q; q)_{n+k}}{(q; q)_{n-m+k} (q^{n+m}; q)_{n+k+1} (q^{n+m+1}; q)_{n+k+1}}.$$

Some simple estimations then give

$$\begin{aligned} \left| \int_0^1 \frac{p_{n,m}(x)}{p^{n+m} - x} d_q x \right| &\leq \frac{q^{n(n-1)/2} q^{m(m-1)/2} q^{n+m}}{p^{(n+m)(n+1)} (q; q)_m} \frac{1}{(q^{n+m}; q)_{n+m+2}^2} \sum_{k=0}^m (q^{2n})^k \\ &= \frac{q^{n(n-1)/2} q^{m(m-1)/2} q^{n+m}}{p^{(n+m)(n+1)} (q; q)_m (q^{n+m}; q)_{n+m+2}^2} \frac{1 - q^{2n(m+1)}}{1 - q^{2n}}. \end{aligned}$$

This gives a useful estimate for the integral on the right hand side of equation (4.5), which (for $m = n - 1$) implies that

$$|\beta_n \zeta_q(1) - \alpha_n| < d_{2n-1}(p) \frac{q^{n(n-1)/2} q^{(n-1)(n-2)/2}}{p^{(2n-1)(n+1)}} \frac{q^{2n-1}}{(q; q)_{n-1} (q^{2n-1}; q)_{2n+1}^2 (1 - q^{2n})}. \quad (4.9)$$

Now we can prove Theorem 1.1

Proof of Theorem 1.1. From Section 3 we know that α_n and β_n are integers. From the equations (4.5) and (4.8) it follows that $\beta_n \zeta_q(1) - \alpha_n \neq 0$ because $d_{2n-1}(p) \neq 0$ and because the integral on the right hand side of (4.5) can be written as a sum with all terms different from zero and of the same sign. From (4.9) we can find that

$$\begin{aligned} \lim_{n \rightarrow \infty} |\beta_n \zeta_q(1) - \alpha_n|^{1/n^2} &\leq \lim_{n \rightarrow \infty} d_{2n-1}(p)^{1/n^2} q^3 \\ &= p^{12/\pi^2} p^{-3} = p^{-\frac{3(\pi^2-4)}{\pi^2}} < 1, \end{aligned}$$

where we have used Lemma 3.1. Hence $\lim_{n \rightarrow \infty} |\beta_n \zeta_q(1) - \alpha_n| \rightarrow 0$. The irrationality now follows from Lemma 1.1. \square

4.3 Measure of irrationality for $\zeta_q(1)$

Theorem 1.1 gives rational approximations α_n/β_n for $\zeta_q(1)$ that satisfy

$$\left| \zeta_q(1) - \frac{\alpha_n}{\beta_n} \right| = \mathcal{O} \left(\frac{p^{\left(\frac{-3(\pi^2-4)}{\pi^2} + \epsilon \right) n^2}}{\beta_n} \right) \quad (4.10)$$

for every $\epsilon > 0$, with $\beta_n = d_{2n-1}(p)p_{n,n-1}(p^{2n-1})$. We already know the asymptotic behavior of $d_{2n-1}(p)$, so what remains is to find the asymptotic behavior of $p_{n,n-1}(p^{2n-1})$ as $n \rightarrow \infty$.

Since $\{1, x, x^2, \dots, x^{n-1}, \log x, x \log x, x^2 \log x, \dots, x^{m-1} \log x\}$ is a Chebyshev system on $[0, 1]$ whenever $m \leq n$, we know that all the zeros of $p_{n,m}(z)$ are simple and lie in the interval $(0, 1)$. If we call these zeros x_1, x_2, \dots, x_{n+m} then we can write $p_{n,m}(x)$ as

$$p_{n,m}(x) = \kappa_{n,m} \prod_{j=1}^{n+m} (x - x_j).$$

For $|x| > 1$ we have $|x| - 1 \leq |x - x_j| \leq |x| + 1$, hence

$$(|x| - 1)^{n+m} \leq \prod_{j=1}^{n+m} |x - x_j| \leq (|x| + 1)^{n+m},$$

so multiplying by $|\kappa_{n,m}|$ and evaluating at $x = p^{n+m}$ gives

$$|\kappa_{n,m}|(p^{n+m} - 1)^{n+m} \leq |p_{n,m}(p^{n+m})| \leq |\kappa_{n,m}|(p^{n+m} + 1)^{n+m}.$$

For $m = n - 1$ one obtains

$$|\kappa_{n,n-1}|^{1/n^2} (p^{2n-1} - 1)^{\frac{2n-1}{n^2}} \leq |p_{n,n-1}(p^{2n-1})|^{1/n^2} \leq |\kappa_{n,n-1}|^{1/n^2} (p^{2n-1} + 1)^{\frac{2n-1}{n^2}},$$

so we have that

$$\lim_{n \rightarrow \infty} |p_{n,n-1}(p^{2n-1})|^{1/n^2} = p^4 \lim_{n \rightarrow \infty} |\kappa_{n,n-1}|^{1/n^2}.$$

Equation (2.16) implies that the leading coefficient of $p_{n,m}$ is given by

$$\kappa_{n,m} = [p_{n,m}^{(0,0,0)}(0)]^{-1} = (-1)^{n+m} \frac{(q^{n+m+1}; q)_n (q^{n+m+1}; q)_m}{q^{nm} (q; q)_n (q; q)_m q^{n(n-1)/2} q^{m(m-1)/2}}. \quad (4.11)$$

Since $(q^{n+1}; q)_k = (q; q)_{n+k} / (q; q)_n$, we can rewrite the leading coefficient of $p_{n,m}$ as

$$\kappa_{n,m} = (-1)^{n+m} \begin{bmatrix} 2n+m \\ n \end{bmatrix}_q \begin{bmatrix} n+2m \\ m \end{bmatrix}_q p^{(n+m)^2/2} p^{-(n+m)/2}. \quad (4.12)$$

This gives $\lim_{n \rightarrow \infty} |\kappa_{n,n-1}|^{1/n^2} = p^2$, so we have

$$\lim_{n \rightarrow \infty} |p_{n,n-1}(p^{2n-1})|^{1/n^2} = p^6. \quad (4.13)$$

Combining this with Lemma 3.1 we have for the integers β_n in (4.3) that

$$\lim_{n \rightarrow \infty} |\beta_n|^{1/n^2} = p^{12/\pi^2} p^6 = p^{\frac{6(\pi^2+2)}{\pi^2}}.$$

Together with (4.10) this gives that

$$\left| \zeta_q(1) - \frac{\alpha_n}{\beta_n} \right| = \mathcal{O} \left(\frac{1}{\beta_n^{1 + \frac{\pi^2 - 4}{2(\pi^2 + 2)} - \epsilon}} \right) \quad (4.14)$$

for every $\epsilon > 0$, which implies the following upper bound for the measure of irrationality

$$r(\zeta_q(1)) \leq 1 + \frac{2(\pi^2 + 2)}{\pi^2 - 4} = \frac{3\pi^2}{\pi^2 - 4} \approx 5.04443.$$

5 Rational approximations to $\zeta_q(2)$

From (1.1) we know that

$$\zeta_q(2) = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{n}{p^n - 1}.$$

In this section we will construct rational approximations to $\zeta_q(2)$, we will prove that $\zeta_q(2)$ is irrational, and we will give an upper bound for its measure of irrationality.

5.1 Rational approximations

Evaluating the function f_2 , which we defined in (3.2), at the point p^{n+m} , gives

$$f_2(p^{n+m}) = \sum_{k=0}^{\infty} \frac{kq^k}{p^{n+m} - q^k} = \sum_{k=0}^{\infty} \frac{k}{p^{n+m+k} - 1}.$$

So we have

$$\begin{aligned} f_2(p^{n+m}) &= \sum_{k=0}^{\infty} \frac{n+m+k}{p^{n+m+k} - 1} - (n+m) \sum_{k=0}^{\infty} \frac{1}{p^{n+m+k} - 1} \\ &= \zeta_q(2) - \sum_{k=1}^{n+m-1} \frac{k}{p^k - 1} - (n+m)f_1(p^{n+m}), \end{aligned}$$

where we used (4.1) for $f_1(p^{n+m})$. Hence we can write $\zeta_q(2)$ as follows:

$$\zeta_q(2) = f_2(p^{n+m}) + \sum_{k=1}^{n+m-1} \frac{k}{p^k - 1} + (n+m)f_1(p^{n+m}). \quad (5.1)$$

We now take $m = n - 1$ and define

$$b_n = d_{2n-1}^2(p) p_{n,n-1}(p^{2n-1}), \quad (5.2)$$

$$a_n = d_{2n-1}^2(p) \left[r_{n,n-1}(p^{2n-1}) + p_{n,n-1}(p^{2n-1}) \sum_{k=1}^{2n-2} \frac{k}{p^k - 1} + (2n-1) q_{n,n-1}(p^{2n-1}) \right]. \quad (5.3)$$

Then it follows from (5.1) and from the Hermite-Padé approximation of f_1 and f_2 (3.8)–(3.9) that

$$b_n \zeta_q(2) - a_n = d_{2n-1}^2(p) \left[\int \frac{p_{n,n-1}(x)}{p^{2n-1} - x} d\mu_2(x) + (2n-1) \int \frac{p_{n,n-1}(x)}{p^{2n-1} - x} d\mu_1(x) \right]. \quad (5.4)$$

From Section 3 we know that the numbers a_n and b_n are integers for all $n \in \mathbb{N}$. We will show that

$$\lim_{n \rightarrow \infty} |b_n \zeta_q(2) - a_n| = 0,$$

and that $b_n \zeta_q(2) - a_n \neq 0$ for all $n \in \mathbb{N}$, so that Lemma 1.1 implies the irrationality of $\zeta_q(2)$. But before we can do that, we have to prove some results about the asymptotic behavior of $p_{n,m}$ as $n, m \rightarrow \infty$.

5.2 Asymptotic behavior of $p_{n,m}$

It is well known that the common denominator of Hermite-Padé approximants satisfies a multiple orthogonality relation. Driver and Stahl [5] showed that for a Nikishin system these common denominators also satisfy an ordinary orthogonality relation. Although in our case (f_1, f_2) do not form a Nikishin system, we can prove in a similar way that $p_{n,m}$ in our case also satisfies an ordinary orthogonality relation. We need the following theorem.

Theorem 5.1 *Let $p_{n,m}$ be the multiple little q -Jacobi polynomial given by (2.14)–(2.15), let $q_{n,m}$ be defined by (3.10) and let $m \leq n$. Then we have*

$$\int_{-\infty}^0 (y-1)^k [p_{n,m}(y) f_1(y) - q_{n,m}(y)] dy = 0 \quad (5.5)$$

for $k = 0, 1, \dots, m-1$.

Proof. From the expression of the remainder of the Hermite-Padé approximation (3.8) it follows that

$$\int_{-\infty}^0 (y-1)^k [p_{n,m}(y) f_1(y) - q_{n,m}(y)] dy = \int_{-\infty}^0 \frac{(y-1)^{k+1}}{y-1} \left[\int_0^1 \frac{p_{n,m}(x)}{y-x} d\mu_1(x) \right] dy.$$

If we add and subtract $(x-1)^{k+1}$ on the right hand side, then we have

$$\begin{aligned}
& \int_{-\infty}^0 (y-1)^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] dy \\
&= \int_{-\infty}^0 \frac{1}{y-1} \left[\int_0^1 \frac{((y-1)^{k+1} - (x-1)^{k+1})}{y-x} p_{n,m}(x) d\mu_1(x) \right] dy \\
&+ \int_{-\infty}^0 \frac{1}{y-1} \left[\int_0^1 \frac{(x-1)^{k+1}}{y-x} p_{n,m}(x) d\mu_1(x) \right] dy.
\end{aligned}$$

Now $((y-1)^{k+1} - (x-1)^{k+1})/(y-x)$ is a polynomial of degree k in x , so because of the orthogonality (2.14) the first integral on the right hand side vanishes for $k < n$. Since $m \leq n$, this integral therefore vanishes for $k = 0, 1, \dots, m-1$. Changing the order of integration gives

$$\int_{-\infty}^0 (y-1)^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] dy = \int_0^1 (x-1)^k p_{n,m}(x) \int_{-\infty}^0 \frac{(x-1)}{(y-1)(y-x)} dy d\mu_1(x).$$

By a partial fraction decomposition we obtain

$$\frac{1}{1-y} - \frac{1}{x-y} = \frac{(x-1)}{(y-1)(y-x)},$$

so we find that

$$\int_{-\infty}^0 \frac{(x-1)}{(y-1)(y-x)} dy = \lim_{t \rightarrow -\infty} \left[\int_t^0 \frac{dy}{1-y} - \int_t^0 \frac{dy}{x-y} \right] = \log x.$$

Using this in the previous expression, we find

$$\int_{-\infty}^0 (y-1)^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] dy = \int_0^1 (x-1)^k p_{n,m}(x) \log x d\mu_1(x).$$

The orthogonality relations (2.15) then imply that the right hand side is equal to zero for $k < m$. \square

From this theorem it follows that $p_{n,m}(y)f_1(y) - q_{n,m}(y)$ has at least m sign changes on the interval $(-\infty, 0)$. For suppose $p_{n,m}(y)f_1(y) - q_{n,m}(y)$ has only $r < m$ sign changes on $(-\infty, 0)$, say at s_1, s_2, \dots, s_r , then we have with $\pi_r(z) = (z-s_1)(z-s_2)\dots(z-s_r)$ that

$$\int_{-\infty}^0 \pi_r(y) [p_{n,m}(y)f_1(y) - q_{n,m}(y)] dy \neq 0$$

because the integrand has no sign changes on $(-\infty, 0)$. This gives a contradiction with the fact that this integral can be written as a linear combination of integrals of the form

$$\int_{-\infty}^0 (y-1)^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] dy$$

with $k \leq r < m$, because from Theorem 5.1 we know that these integrals are all zero. So $p_{n,m}(y)f_1(y) - q_{n,m}(y)$ has at least m sign changes on $(-\infty, 0)$. The condition $m \leq n$ and (3.3) guarantees that these integrals are finite.

Let s_1, s_2, \dots, s_m be m points where $p_{n,m}(z)f_1(z) - q_{n,m}(z)$ changes sign on $(-\infty, 0)$. Then we define the polynomial w by

$$w(z) = (z - s_1)(z - s_2) \dots (z - s_m). \quad (5.6)$$

Now we can prove that $p_{n,m}$ also satisfies an ordinary orthogonality relation.

Theorem 5.2 *Let w be defined by (5.6) and let $p_{n,m}$ the multiple little q -Jacobi polynomial defined by (2.14)–(2.15). Then*

$$\int_0^1 p_{n,m}(y) y^k \frac{d\mu_1(y)}{w(y)} = 0,$$

for $k = 0, 1, \dots, n + m - 1$.

Proof. Let γ be a closed positively oriented path of integration with winding number 1 for all its interior points, such that the interval $[0, 1]$ is in the interior of γ and the zeros of w are outside γ . Using (3.8) we have that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{y^k}{w(y)} \int_0^1 \frac{p_{n,m}(x)}{x - y} d\mu_1(x) dy = \frac{1}{2\pi i} \int_{\gamma} y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] \frac{dy}{w(y)}.$$

Changing the order of integration on the left hand side gives

$$\int_0^1 p_{n,m}(x) \frac{1}{2\pi i} \int_{\gamma} \frac{y^k}{x - y} \frac{dy}{w(y)} d\mu_1(x) = \frac{1}{2\pi i} \int_{\gamma} y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] \frac{dy}{w(y)}.$$

The only singularity inside γ is x , hence Cauchy's formula on the left hand side gives

$$\int_0^1 p_{n,m}(x) \frac{x^k}{w(x)} d\mu_1(x) = \frac{1}{2\pi i} \int_{\gamma} y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] \frac{dy}{w(y)}.$$

Since all zeros of w are also zeros of $p_{n,m}(y)f_1(y) - q_{n,m}(y)$, the function $y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)]/w(y)$ is analytic in $\mathbb{C} \setminus [0, 1]$. Furthermore γ encloses the interval $[0, 1]$, hence for R sufficient large

$$\left| \int_0^1 p_{n,m}(x) x^k \frac{d\mu_1(x)}{w(x)} \right| = \left| \frac{1}{2\pi i} \int_{\Gamma_R} y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)] \frac{dy}{w(y)} \right|,$$

with Γ_R the circle with center 0 and radius R . We can estimate the expression by

$$\left| \int_0^1 p_{n,m}(x) x^k \frac{d\mu_1(x)}{w(x)} \right| \leq R \max_{y \in \Gamma_R} \left| y^k \frac{p_{n,m}(y)f_1(y) - q_{n,m}(y)}{w(y)} \right|.$$

Using (3.3), we know that the function $y^k [p_{n,m}(y)f_1(y) - q_{n,m}(y)]/w(y)$ has a zero of order $n + m + 1 - k$ at infinity. So when R tends to infinity we find

$$\left| \int_0^1 p_{n,m}(x) x^k \frac{d\mu_1(x)}{w(x)} \right| = \mathcal{O} \left(\frac{1}{R^{n+m-k}} \right).$$

Therefore

$$\int_0^1 p_{n,m}(x) x^k \frac{d\mu_1(x)}{w(x)} = 0$$

when $k \leq n + m - 1$. □

Now we have proved that $p_{n,m}$ satisfies an ordinary orthogonality relation. Using this, we can easily show that $p_{n,m}(z)f_1(z) - q_{n,m}(z)$ has exactly m sign changes in the interval $(-\infty, 0)$, namely at the zeros of w . Using (3.8) we have

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = \int_0^1 \frac{p_{n,m}(y)}{z-y} \frac{w(y)}{w(y)} d\mu_1(y).$$

If we add and subtract $w(z)$ on the right hand side, then we get

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = \int_0^1 \frac{p_{n,m}(y)}{z-y} \frac{w(y) - w(z)}{w(y)} d\mu_1(y) + w(z) \int_0^1 \frac{p_{n,m}(y)}{z-y} \frac{d\mu_1(y)}{w(y)}.$$

The first integral on the right hand side vanishes because of the orthogonality (Theorem 5.2), so that

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = w(z) \frac{p_{n,m}(z)}{p_{n,m}(z)} \int_0^1 \frac{p_{n,m}(y)}{z-y} \frac{d\mu_1(y)}{w(y)}.$$

Now we add and subtract $p_{n,m}(y)$ on the right hand side, so we get

$$\begin{aligned} p_{n,m}(z)f_1(z) - q_{n,m}(z) = \\ \frac{w(z)}{p_{n,m}(z)} \int_0^1 \frac{p_{n,m}(y)}{z-y} (p_{n,m}(z) - p_{n,m}(y)) \frac{d\mu_1(y)}{w(y)} + \frac{w(z)}{p_{n,m}(z)} \int_0^1 \frac{p_{n,m}^2(y)}{z-y} \frac{d\mu_1(y)}{w(y)}. \end{aligned}$$

Again, the first expression on the right hand side vanishes because of the orthogonality (Theorem 5.2), so we find

$$p_{n,m}(z)f_1(z) - q_{n,m}(z) = \frac{w(z)}{p_{n,m}(z)} \int_0^1 \frac{p_{n,m}^2(y)}{z-y} \frac{d\mu_1(y)}{w(y)}. \quad (5.7)$$

The integrand is not identically zero and has a constant sign for $z \in \mathbb{R} \setminus [0, 1]$, hence the integral can not be zero, so it follows from this expression that the sign changes of $p_{n,m}(z)f_1(z) - q_{n,m}(z)$ in $\mathbb{R} \setminus [0, 1]$ are at the m simple zeros of w in $(-\infty, 0)$.

5.3 Irrationality of $\zeta_q(2)$

We will now show that $\lim_{n \rightarrow \infty} (b_n \zeta_q(2) - a_n) = 0$ and $b_n \zeta_q(2) - a_n \neq 0$ for all $n \in \mathbb{N}$, so that Lemma 1.1 implies the irrationality of $\zeta_q(2)$. First we can reduce the right hand side of (5.4) as follows. When we look at the Cauchy transform of $p_{n,m}(z)f_1(z) - q_{n,m}(z)$ and

use (3.8), then we find

$$\int_{-\infty}^0 \frac{p_{n,m}(z)f_1(z) - q_{n,m}(z)}{y - z} dz = \int_{-\infty}^0 \frac{dz}{y - z} \int_0^1 \frac{p_{n,m}(x)}{z - x} d\mu_1(x).$$

Because

$$\frac{1}{(y - z)(z - x)} = \frac{1}{(y - x)} \left[\frac{1}{y - z} - \frac{1}{x - z} \right],$$

we find by changing the order of integration

$$\begin{aligned} \int_{-\infty}^0 \frac{p_{n,m}(z)f_1(z) - q_{n,m}(z)}{y - z} dz &= \int_0^1 \frac{p_{n,m}(x)}{y - x} d\mu_1(x) \int_{-\infty}^0 \left[\frac{1}{y - z} - \frac{1}{x - z} \right] dz \\ &= \int_0^1 \frac{p_{n,m}(x)}{y - x} \log x d\mu_1(x) - \log y \int_0^1 \frac{p_{n,m}(x)}{y - x} d\mu_1(x). \end{aligned} \quad (5.8)$$

By multiplying both sides by $d_{n+m}^2(p)/\log q$ and using $\log x d\mu_1(x)/\log q = \log_q x d\mu_1(x) = d\mu_2(x)$, we obtain

$$\begin{aligned} \frac{d_{n+m}^2(p)}{\log q} \int_{-\infty}^0 \frac{p_{n,m}(z)f_1(z) - q_{n,m}(z)}{y - z} dz \\ = d_{n+m}^2(p) \left[\int_0^1 \frac{p_{n,m}(x)}{y - x} d\mu_2(x) - \frac{\log y}{\log q} \int_0^1 \frac{p_{n,m}(x)}{y - x} d\mu_1(x) \right]. \end{aligned}$$

Evaluating at $z = p^{n+m}$ gives the right hand side of (5.4) when $m = n - 1$, so it turns out that

$$b_n \zeta_q(2) - a_n = \frac{d_{2n-1}^2(p)}{\log q} \int_{-\infty}^0 \frac{p_{n,n-1}(z)f_1(z) - q_{n,n-1}(z)}{p^{2n-1} - z} dz. \quad (5.9)$$

Now we will estimate the expression on the right hand side. Multiplying and dividing by $w(p^{n+m})$ and adding and subtracting $w(z)$ gives that the right hand side is equal to

$$\begin{aligned} \frac{d_{2n-1}^2(p)}{\log q} \frac{1}{w(p^{2n-1})} \left[\int_{-\infty}^0 (w(p^{2n-1}) - w(z)) \frac{p_{n,n-1}(z)f_1(z) - q_{n,n-1}(z)}{p^{2n-1} - z} dz \right. \\ \left. + \int_{-\infty}^0 w(z) \frac{p_{n,n-1}(z)f_1(z) - q_{n,n-1}(z)}{p^{2n-1} - z} dz \right]. \end{aligned}$$

The first term is 0 because of the orthogonality relations given by Theorem 5.1, so we find

$$b_n \zeta_q(2) - a_n = \frac{d_{2n-1}^2(p)}{\log q} \frac{1}{w(p^{2n-1})} \int_{-\infty}^0 w(z) \frac{p_{n,n-1}(z)f_1(z) - q_{n,n-1}(z)}{p^{2n-1} - z} dz. \quad (5.10)$$

Since w is a monic polynomial of degree $m = n - 1$ and all the zeros of w are in the interval $(-\infty, 0)$, we have that $w(p^{2n-1}) > (p^{2n-1})^{n-1}$. We also have that $p^{2n-1} - z > p^{2n-1}$ for $z \in (-\infty, 0)$, so we find

$$|b_n \zeta_q(2) - a_n| \leq -\frac{d_{2n-1}^2(p)}{\log q} (q^{2n-1})^n \left| \int_{-\infty}^0 w(z) [p_{n,n-1}(z)f_1(z) - q_{n,n-1}(z)] dz \right|. \quad (5.11)$$

The integral on the right hand side can be evaluated exactly, which is done in the following lemma.

Lemma 5.1 *Suppose that $m \leq n - 1$, then*

$$\int_{-\infty}^0 w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)]dz = q^{n(m+1)}(q; q)_m \frac{(q; q)_{n-m-1}}{(q^{n+m+1}; q)_{m+1}} \frac{(q; q)_m}{(q; q)_n} \log q. \quad (5.12)$$

Proof. Multiply both sides of equation (5.8) by $w(y)$, then add and subtract $w(z)$ on both sides of this expression and use the orthogonality relations (5.5) for the left hand side and (2.14)–(2.15) for the right hand side, to find

$$\begin{aligned} \int_{-\infty}^0 \frac{w(z)}{y-z} [p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz = \\ \int_0^1 \frac{w(z)}{y-z} p_{n,m}(z) \log z d\mu_1(z) - \log y \int_0^1 \frac{w(z)}{y-z} p_{n,m}(z) d\mu_1(z). \end{aligned}$$

Multiply both sides by y and add and subtract z in the second term on the right hand side to find

$$\begin{aligned} \int_{-\infty}^0 \frac{y}{y-z} w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz = \int_0^1 \frac{y}{y-z} w(z) p_{n,m}(z) \log z d\mu_1(z) \\ - \log y \int_0^1 w(z) p_{n,m}(z) d\mu_1(z) - \log y \int_0^1 \frac{z}{y-z} w(z) p_{n,m}(z) d\mu_1(z). \end{aligned}$$

We know that w is a polynomial of degree m . For $m \leq n - 1$ the orthogonality relations (2.14)–(2.15) imply that the second integral on the right hand side is 0. Taking the limit for y going to ∞ , we get

$$\int_{-\infty}^0 w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz = \int_0^1 w(z) p_{n,m}(z) \log z d\mu_1(z).$$

Since w is a monic polynomial of degree m , it follows from the orthogonality relations (2.15) that

$$\int_{-\infty}^0 w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz = \int_0^1 z^m p_{n,m}(z) \log z d\mu_1(z). \quad (5.13)$$

Using the Rodrigues formula (2.18) for $p_{n,m}$ on the right hand side we get

$$\int_{-\infty}^0 w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz = C \int_0^1 z^m \log z D_{q^{-1}}^n [z^n D_{q^{-1}}^m ((qz; q)_{n+m} z^m)] d\mu_1(z),$$

with C given by expression (4.6). Repeated application of summation by parts (2.11), gives

$$\begin{aligned} \int_{-\infty}^0 w(z)[p_{n,m}(z)f_1(z) - q_{n,m}(z)] dz \\ = (-1)^n q^n C \int_0^1 D_q^n [z^m \log z] z^n D_{q^{-1}}^m [(qz; q)_{n+m} z^m] d\mu_1(z). \end{aligned}$$

For $D_q^n [z^m \log z]$ we use the q -analogue of Leibniz' rule (2.12) to find

$$D_q^n [z^m \log z] = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} x^m \right) \Big|_{x=q^k z} \left(D_q^k \log x \right) \Big|_{x=z}. \quad (5.14)$$

Equation (4.7) gives us an expression for $D_q^{n-k} z^m$ when $n - k \leq m$. For $n - k > m$ we know that $D_q^{n-k} z^m$ is 0. It is also easy to see that for $k > 0$

$$D_q^k \log x = (-1)^k q^{-k(k-1)/2} \frac{\log q}{(1-q)^k} \frac{(q; q)_{k-1}}{x^k},$$

so it turns out that

$$\begin{aligned} \int_{-\infty}^0 w(z) [p_{n,m}(z) f_1(z) - q_{n,m}(z)] dz &= (-1)^n q^n C \sum_{k=n-m}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q; q)_m}{(1-q)^n} \frac{(q^k)^{m-n+k}}{(q; q)_{m-n+k}} \\ &\times (-1)^k q^{-k(k-1)/2} (q; q)_{k-1} \log q \int_0^1 z^m D_{q^{-1}}^m [(qz; q)_{n+m} z^m] d\mu_1(z). \end{aligned} \quad (5.15)$$

We now have to evaluate a q -integral and a finite sum.

(1) We start with the integral in (5.15). Apply summation by parts m times to find

$$\int_0^1 z^m D_{q^{-1}}^m [(qz; q)_{n+m} z^m] d\mu_1(z) = (-1)^m q^m \int_0^1 D_q^m [z^m] (qz; q)_{n+m} z^m d\mu_1(z).$$

When we use equation (4.7) for $D_q^m [z^m]$ and rewrite the q -integral as a infinite sum using (2.1), it turns out that

$$\int_0^1 z^m D_{q^{-1}}^m [(qz; q)_{n+m} z^m] d\mu_1(z) = (-1)^m q^m \frac{(q; q)_m}{(1-q)^m} \sum_{j=0}^{\infty} (q^j)^{m+1} (q^{j+1}; q)_{n+m}.$$

We can compute the sum on the right hand side by using the q -binomial series (2.21). This gives

$$\begin{aligned} \sum_{j=0}^{\infty} (q^{m+1})^j (q^{j+1}; q)_{n+m} &= (q; q)_{n+m} \sum_{j=0}^{\infty} (q^{m+1})^j \frac{(q^{n+m+1}; q)_j}{(q; q)_j} \\ &= (q; q)_{n+m} \frac{(q^{n+2m+2}; q)_{\infty}}{(q^{m+1}; q)_{\infty}} \\ &= \frac{(q; q)_m}{(q^{n+m+1}; q)_{m+1}}. \end{aligned}$$

So we find that

$$\int_0^1 z^m D_{q^{-1}}^m [(qz; q)_{n+m} z^m] d\mu_1(z) = (-1)^m q^m \frac{(q; q)_m}{(1-q)^m} \frac{(q; q)_m}{(q^{n+m+1}; q)_{m+1}}. \quad (5.16)$$

(2) Now we will work out the finite sum

$$\sum_{k=n-m}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q; q)_m}{(1-q)^n} \frac{(q^k)^{m-n+k}}{(q; q)_{m-n+k}} (-1)^k q^{-k(k-1)/2} (q; q)_{k-1}$$

in (5.15). Changing the index of summation gives

$$\sum_{k=0}^m \begin{bmatrix} n \\ k+n-m \end{bmatrix}_q \frac{(q; q)_m}{(1-q)^n} \frac{(q^{k+n-m})^k}{(q; q)_k} (-1)^{k+n-m} q^{-(k+n-m)(k+n-m-1)/2} (q; q)_{k+n-m-1}.$$

Writing out the q -binomial coefficient and rewriting the powers of q gives that this is equal to

$$\sum_{k=0}^m (-1)^{k+n-m} \frac{(q; q)_n (q; q)_m (q; q)_{k+n-m-1}}{(q; q)_{k+n-m} (q; q)_{m-k} (q; q)_k} \frac{q^{k(k+1)/2} q^{mn} q^{-n(n-1)/2} q^{-m(m+1)/2}}{(1-q)^n}.$$

Use $(q; q)_{k+n-m-1}/(q; q)_{k+n-m} = 1/(1-q^{k+n-m})$ to rewrite this sum as

$$(-1)^{n+m} q^{mn} q^{-n(n-1)/2} q^{-m(m+1)/2} \frac{(q; q)_n}{(1-q)^n} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{q^{k(k+1)/2}}{(1-q^{k+n-m})}.$$

Now use

$$\frac{1}{1-q^{k+n-m}} = \sum_{\ell=0}^{\infty} (q^{k+n-m})^{\ell}$$

to find

$$(-1)^{n+m} q^{mn} q^{-n(n-1)/2} q^{-m(m+1)/2} \frac{(q; q)_n}{(1-q)^n} \sum_{\ell=0}^{\infty} (q^{n-m})^{\ell} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q (q^k)^{\ell+1} q^{k(k-1)/2}.$$

If we now use the q -analog of Newton's binomial formula (2.19) with $x = q^{\ell+1}$ then the expression reduces to

$$(-1)^{n+m} q^{mn} q^{-n(n-1)/2} q^{-m(m+1)/2} \frac{(q; q)_n}{(1-q)^n} \sum_{\ell=0}^{\infty} (q^{n-m})^{\ell} (q^{\ell+1}; q)_m.$$

Using the q -binomial series (2.21), we have

$$\begin{aligned} \sum_{\ell=0}^{\infty} (q^{n-m})^{\ell} (q^{\ell+1}; q)_m &= (q; q)_m \sum_{\ell=0}^{\infty} (q^{n-m})^{\ell} \frac{(q^{m+1}; q)_{\ell}}{(q; q)_{\ell}} \\ &= (q; q)_m \frac{(q^{n+1}; q)_{\infty}}{(q^{n-m}; q)_{\infty}} \\ &= (q; q)_{n-m-1} \frac{(q; q)_m}{(q; q)_n}. \end{aligned}$$

With this we then find

$$\begin{aligned} \sum_{k=n-m}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q; q)_m}{(1-q)^n} \frac{(q^k)^{m-n+k}}{(q; q)_{m-n+k}} (-1)^k q^{-k(k-1)/2} (q; q)_{k-1} \\ = (-1)^{n+m} q^{mn} q^{-n(n-1)/2} q^{-m(m+1)/2} \frac{(q; q)_m}{(1-q)^n} (q; q)_{n-m-1}. \end{aligned} \quad (5.17)$$

Using this result together with (5.16) and the expression (4.6) for C , we find that (5.15) gives the required result. \square

The integral in (5.12) can also be evaluated for $m = n$ but with much more effort, which is the main reason why we have chosen $m = n - 1$ for our rational approximants. Lemma 5.1 and (5.11) now imply that for $m = n - 1$

$$|b_n \zeta_q(2) - a_n| \leq d_{2n-1}^2(p) \left(q^{2n-1} \right)^n q^{n^2} \frac{(q; q)_{n-1}^2}{(q^{2n}; q)_n (q; q)_n}. \quad (5.18)$$

Now we can prove Theorem 1.2

Proof of Theorem 1.2. From Section 3 we know that a_n and b_n are integers. From (5.10) it follows that $b_n \zeta_q(2) - a_n \neq 0$ because the integrand has a constant sign. Furthermore Lemma 3.1 gives that

$$\lim_{n \rightarrow \infty} \left(d_{2n-1}^2(p) \right)^{1/n^2} = p^{24/\pi^2}.$$

From (5.18) we see that

$$\lim_{n \rightarrow \infty} |b_n \zeta_q(2) - a_n|^{1/n^2} \leq p^{\frac{24}{\pi^2}} \lim_{n \rightarrow \infty} q^{\frac{2n-1}{n}} q = p^{\frac{24}{\pi^2}} q^3 = p^{-\frac{3(\pi^2-8)}{\pi^2}} < 1$$

so that $\lim_{n \rightarrow \infty} |b_n \zeta_q(2) - a_n| \rightarrow 0$ for $n \rightarrow \infty$. The irrationality now follows from Lemma 1.1. \square

5.4 Measure of irrationality for $\zeta_q(2)$

Theorem 1.2 gives rational approximations a_n/b_n for $\zeta_q(2)$ that satisfy

$$\left| \zeta_q(2) - \frac{a_n}{b_n} \right| = \mathcal{O} \left(\frac{p^{\left(\frac{-3(\pi^2-8)}{\pi^2} + \epsilon \right) n^2}}{b_n} \right) \quad (5.19)$$

for every $\epsilon > 0$, with $b_n = d_{2n-1}^2(p) p_{n,n-1} (p^{2n-1})$. Combining the result of Lemma 3.1 with the asymptotic behavior of $p_{n,n-1} (p^{2n-1})$ given in (4.13) gives

$$\lim_{n \rightarrow \infty} |b_n|^{1/n^2} = \lim_{n \rightarrow \infty} \left| d_{2n-1}^2(p) p_{n,n-1} (p^{2n-1}) \right|^{1/n^2} = p^{24/\pi^2} p^6 = p^{\frac{6(\pi^2+4)}{\pi^2}}.$$

Together with (5.19) this gives

$$\left| \zeta_q(2) - \frac{a_n}{b_n} \right| = \mathcal{O} \left(\frac{1}{b_n^{1 + \frac{\pi^2 - 8}{2(\pi^2 + 4)} - \epsilon}} \right) \quad (5.20)$$

for every $\epsilon > 0$, which implies the following upper bound for the measure of irrationality

$$r(\zeta_q(2)) \leq 1 + \frac{2(\pi^2 + 4)}{\pi^2 - 8} = \frac{3\pi^2}{\pi^2 - 8} \approx 15.8369.$$

6 Linear independence of $1, \zeta_q(1), \zeta_q(2)$ over \mathbb{Q}

In this section we prove Theorem 1.3 by using Lemma 1.2. To apply this lemma we need rational approximations for $\zeta_q(1)$ and $\zeta_q(2)$ with common denominator. From the equations (4.3)-(4.4) and (5.2)-(5.3), it follows that we can take $p_n^* = b_n = d_{2n-1}(p)\beta_n$, $q_n^* = d_{2n-1}(p)\alpha_n$ and $r_n^* = a_n$, i.e.

$$p_n^* = d_{2n-1}^2(p) p_{n,n-1}(p^{2n-1}) \quad (6.1)$$

$$q_n^* = d_{2n-1}^2(p) \left[q_{n,n-1}(p^{2n-1}) + p_{n,n-1}(p^{2n-1}) \sum_{k=1}^{2n-2} \frac{1}{p^k - 1} \right] \quad (6.2)$$

$$r_n^* = d_{2n-1}^2(p) \left[r_{n,n-1}(p^{2n-1}) + p_{n,n-1}(p^{2n-1}) \sum_{k=1}^{2n-2} \frac{k}{p^k - 1} + (2n-1)q_{n,n-1}(p^{2n-1}) \right]. \quad (6.3)$$

Then Theorems 1.1 and 1.2 imply that p_n^* , q_n^* and r_n^* are integers. The following verification of the 3 conditions of Lemma 1.2 will establish the linear independence of $1, \zeta_q(1)$ and $\zeta_q(2)$ over \mathbb{Q} .

6.1 Condition 1

We will now show that $ap_n^* + bq_n^* + cr_n^* \neq 0$ for all $n \in \mathbb{N}$ for which $2n-1$, is prime and $2n-1 > c$. It is sufficient to prove that

$$ap_n^* + bq_n^* + cr_n^* \not\equiv 0 \pmod{\Phi_{2n-1}(p)}$$

or

$$\Phi_{2n-1}(p) \nmid ap_n^* + bq_n^* + cr_n^*$$

for these values of n . We prove this in three steps.

Step 1: In the first step we prove that

$$ap_n^* + bq_n^* + cr_n^* \equiv -c \frac{d_{2n-1}^2(p)}{(p^{2n-1} - 1)^2} \pmod{\Phi_{2n-1}(p)}. \quad (6.4)$$

First note that $d_{2n-1}(p) = \Phi_{2n-1}(p)d_{2n-2}(p)$, which implies that $\Phi_{2n-1}(p)$ divides the integers $d_{2n-1}^2(p)p_{n,n-1}(p^{2n-1})$ and $d_{2n-1}^2(p)q_{n,n-1}(p^{2n-1})$ in \mathbb{Z} . Using this we get

$$\begin{aligned} ap_n^* + bq_n^* + cr_n^* &\equiv c d_{2n-1}^2(p)r_{n,n-1}(p^{2n-1}) \pmod{\Phi_{2n-1}(p)} \\ &\equiv c d_{2n-1}(p)d_{2n-2}(p)\Phi_{2n-1}(p)r_{n,n-1}(p^{2n-1}) \pmod{\Phi_{2n-1}(p)}. \end{aligned}$$

Expression (3.18) implies that all the terms in the sum $c d_{2n-1}(p)d_{2n-2}(p)r_{n,n-1}(p^{2n-1})$ are integers except for the one with $r = i = 2n - 1$. So we see

$$ap_n^* + bq_n^* + cr_n^* \equiv -c \begin{bmatrix} 3n-2 \\ n-1 \end{bmatrix}_p \begin{bmatrix} 3n-1 \\ n \end{bmatrix}_p \frac{d_{2n-1}^2(p)}{(p^{2n-1}-1)^2} \pmod{\Phi_{2n-1}(p)}.$$

Now we only have to eliminate the binomial numbers. This can be done by the following lemma. By applying it twice, with (n, m) replaced by $(n, n-1)$ and $(n-1, n)$, we obtain equation (6.4).

Lemma 6.1 *The following congruence for polynomials in $\mathbb{Z}[x]$ holds*

$$\begin{bmatrix} n+2m \\ m \end{bmatrix}_x \equiv 1 \pmod{\Phi_{n+m}(x)} \quad (6.5)$$

for all $n, m \in \mathbb{N}$.

Proof. We prove this result by induction on m . Obviously relation (6.5) is satisfied for all $n \in \mathbb{N}$ when $m = 0$. Suppose that

$$\begin{bmatrix} n+2m-2 \\ m-1 \end{bmatrix}_x \equiv 1 \pmod{\Phi_{n+m-1}(x)} \quad (6.6)$$

for all $n \in \mathbb{N}$, then we can prove (6.5) as follows. The q -version of Pascal's triangle identity (3.16) gives that

$$\begin{bmatrix} n+2m \\ m \end{bmatrix}_x = \begin{bmatrix} n+2m-1 \\ m \end{bmatrix}_x + x^{n+m} \begin{bmatrix} n+2m-1 \\ m-1 \end{bmatrix}_x.$$

Note that (3.21) implies that $x^{n+m} \equiv 1 \pmod{\Phi_{n+m}(x)}$. We can also write, using (3.21),

$$\begin{bmatrix} n+2m-1 \\ m \end{bmatrix}_x = \frac{(x; x)_{n+2m-1}}{(x; x)_m (x; x)_{n+m-1}} = \frac{\prod_{\nu=n+m}^{n+2m-1} (1-x^\nu)}{\prod_{\nu=1}^m (1-x^\nu)} = \frac{\prod_{\nu=n+m}^{n+2m-1} \prod_{d|\nu} \Phi_d(x)}{\prod_{\nu=1}^m \prod_{d|\nu} \Phi_d(x)}.$$

Since $\begin{bmatrix} n+2m-1 \\ m \end{bmatrix}_x$ is a polynomial in x with integer coefficients, the cyclotomic polynomials $\Phi_d(x)$ are irreducible over \mathbb{Q} , and $\Phi_{n+m}(x)$ is a factor of the numerator and not of the

denominator, it follows that $\Phi_{n+m}(x)$ divides $\left[\frac{n+2m-1}{m} \right]_x$. Using this, it turns out that

$$\left[\frac{n+2m}{m} \right]_x \equiv \left[\frac{n+2m-1}{m-1} \right]_x \pmod{\Phi_{n+m}(x)}.$$

Now we can apply the induction hypothesis (6.6) with $n+1$ instead of n . This proves the lemma. \square

Step 2: In the second step we prove that

$$\gcd(c, \Phi_{n+m}(p)) = 1. \quad (6.7)$$

For this we need the following result of Legendre (see, e.g. [7])

Lemma 6.2 *For all positive integers p, s and every cyclotomic polynomial Φ_n we have that if $s \mid \Phi_n(p)$ then $s = n\ell + 1$ for some $\ell \in \mathbb{N}_0$, or $s \mid n$.*

Suppose $\gcd(c, \Phi_{2n-1}(p)) = s > 1$. Then $s \mid \Phi_{2n-1}(p)$ so from Lemma 6.2 it follows that $s = \ell(2n-1) + 1$ for some $\ell \in \mathbb{N}_0$ or $s \mid 2n-1$. Since $s > 1$ and $2n-1$ is a prime number, we have in both cases that $s \geq 2n-1$. But $s \mid c$ and we only consider values of n for which $2n-1 > c$. This gives a contradiction and proves equation (6.7).

Step 3: In the last step we prove that

$$\Phi_{2n-1}(p) \nmid \frac{d_{2n-1}^2(p)}{(p^{2n-1} - 1)^2}. \quad (6.8)$$

We will do this by contraposition: suppose that there exist an integer $A \neq 0$ such that

$$\frac{d_{2n-1}^2(p)}{(p^{2n-1} - 1)^2} = A \Phi_{2n-1}(p)$$

or, when we use (3.19)–(3.21),

$$\prod_{\substack{k=1 \\ k \nmid 2n-1}}^{2n-1} \Phi_k^2(p) = A \Phi_{2n-1}(p).$$

Suppose s is a prime number so that $s \mid \Phi_{2n-1}(p)$, then it follows from the previous equation that there exists an integer k , with $2 \leq k \leq 2n-2$, such that $s \mid \Phi_k(p)$. We first prove that this implies $\gcd(\Phi_{2n-1}(x), \Phi_k(x)) = 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$. For this we need the following lemma [17, Section 4.9]:

Lemma 6.3 *If s is a prime not dividing n , then $x^n - 1$ has no repeated factors in $(\mathbb{Z}/s\mathbb{Z})[x]$.*

Since $s \mid \Phi_{2n-1}(p)$ we can first apply Lemma 6.2 and distinguish between two cases.

case 1: $s = \ell(2n - 1) + 1$ for some $\ell \in \mathbb{N}_0$. In this case it is obvious that $s \nmid 2n - 1$, and also $s \nmid k$ because $k \leq 2n - 2 < s$. So we can apply Lemma 6.3 with parameter $k(2n - 1)$ instead of n . We get that $x^{k(2n-1)} - 1$ has no repeated factors in $(\mathbb{Z}/s\mathbb{Z})[x]$. This means, using (3.21), that

$$\Phi_k(x) \cdot \Phi_{2n-1}(x) \cdot \prod_{\substack{d|k(2n-1) \\ d \neq k, d \neq 2n-1}} \Phi_d(x)$$

has no repeated factors, so $\gcd(\Phi_{2n-1}(x), \Phi_k(x)) = 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$.

case 2: $s \mid 2n - 1$. Since s and $2n - 1$ are both prime, we see that $s = 2n - 1$ in this case. Since s is prime, we also know that $\Phi_s(x) = (x^s - 1)/(x - 1)$. In $(\mathbb{Z}/s\mathbb{Z})[x]$ this implies

$$\Phi_s(x) = \frac{x^s - 1}{x - 1} = \frac{(x - 1)^s}{x - 1} = (x - 1)^{s-1}.$$

So $\gcd(\Phi_k(x), \Phi_{2n-1}(x)) = \gcd(\Phi_k(x), (x - 1)^{s-1}) = (x - 1)^u$, for some u with $0 \leq u \leq s - 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$. We know that s is a prime not dividing k , so applying Lemma 6.3 gives that $x^k - 1$ has no repeated factors in $(\mathbb{Z}/s\mathbb{Z})[x]$. Now we can write $x^k - 1$ as

$$x^k - 1 = \Phi_1(x) \cdot \Phi_k(x) \cdot \prod_{\substack{d|k \\ 1 < d < k}} \Phi_d(x).$$

and since $\Phi_1(x) = x - 1$, it follows that $x - 1$ can not be a factor of $\Phi_k(x)$, so we can conclude that $\gcd(\Phi_{2n-1}(x), \Phi_k(x)) = 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$.

So in both cases we have $\gcd(\Phi_{2n-1}(x), \Phi_k(x)) = 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$. Integer division gives us two polynomials $u(x)$ and $v(x)$ such that $u(x)\Phi_{2n-1}(x) + v(x)\Phi_k(x) = 1$ in $(\mathbb{Z}/s\mathbb{Z})[x]$. Evaluating this expression at $x = p$ gives in \mathbb{Z} the equality

$$u(p)\Phi_{2n-1}(p) + v(p)\Phi_k(p) = 1 + sw,$$

where w is an integer. But from $s \mid \Phi_{n+m}(p)$ and $s \mid \Phi_k(p)$ we see that $s \mid 1$, which gives a contradiction and proves (6.8).

From these three steps it follows that $\Phi_{2n-1}(p) \nmid ap_n^* + bq_n^* + cr_n^*$ for infinity many values of n , so the first condition of Lemma 1.2 is satisfied.

Remark: For the case $c = 0$, $b \neq 0$ we can work analogously by considering

$$ap_n^* + bq_n^* + cr_n^* \pmod{d_{2n-1}(p)\Phi_{2n-1}(p)}.$$

The case $c = b = 0$ is obvious.

6.2 Condition 2 and 3

From the definition of p_n^* and q_n^* , it follows that $|p_n^* \zeta_q(1) - q_n^*| = d_{2n-1}(p) |\beta_n \zeta_q(1) - \alpha_n|$. By using Theorem 1.1 together with Lemma 3.1, it turns out that

$$|p_n^* \zeta_q(1) - q_n^*|^{1/n^2} = (d_{2n-1}(p))^{1/n^2} |\beta_n \zeta_q(1) - \alpha_n|^{1/n^2} = p^{-\frac{3(\pi^2-8)}{4\pi^2}} < 1,$$

so that $|p_n^* \zeta_q(1) - q_n^*| \rightarrow 0$ when $n \rightarrow \infty$.

It also follows from the definition of p_n^* and r_n^* that $|p_n^* \zeta_q(2) - r_n^*| = |b_n \zeta_q(2) - a_n|$ and from Theorem 1.2 we know already that this expression tends to zero when n tends to infinity.

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